Expected utility model revisited

Remember that our problem is to choose the best option among risky choices $a_1, a_2, \ldots$. So far, we have seen a model to solve it—the expected utility model

$$a^* = \underset{a\in\{a_1,a_2,\ldots\}}{\text{argmax}} \ \mathbb{E}[U(a)].$$

Healthy questions are "Does this work?", "When/Where?" or "Alternative models?". Any theory/model works only where its assumptions hold.

Most alternative models try to relax the linearity in probability

$$\mathbb{E}[U(x)] = \sum_i p_i U(x_i).$$

But, there is no such thing as a free lunch. We must make a fundamental tradeoff in modeling—fidelity to the phenomenon of question and abstraction for ease of analysis. Let's begin with the target of criticism, i.e. the linearity of preference in probability.

Linearity in probability and independence assumption

Given the risky choices a_1, a_2, a_3, for any $\alpha \in [0, 1]$,

$$a_1 \succsim a_2 \quad \Leftrightarrow \quad \alpha a_1 + (1-\alpha)a_3 \succsim \alpha a_2 + (1-\alpha)a_3.$$

In words, taking a convex combination with another choice does not reverse the preference order. Intuitively, mixing another choice is irrelevant (hence, independence).

This assumption is responsible for the linearity of preferences in the space of probabilities, underpinning the expected utility model. The Allais paradox provides a case when it is violated. But really, how are they connected? A key is convexity.

Suppose that a risky choice x takes three possible values: $x_1 < x_2 < x_3$ with probabilities p_1, p_2, and p_3 respectively. Let's write $p_2 = 1 - p_1 - p_3$. Then, if we start with the expected utility,

$$\mathbb{E}[U(x)] = \sum_{i=1}^{3} p_i U(x_i) = p_1 U(x_1) + (1 - p_1 - p_3)U(x_2) + p_3 U(x_3),$$

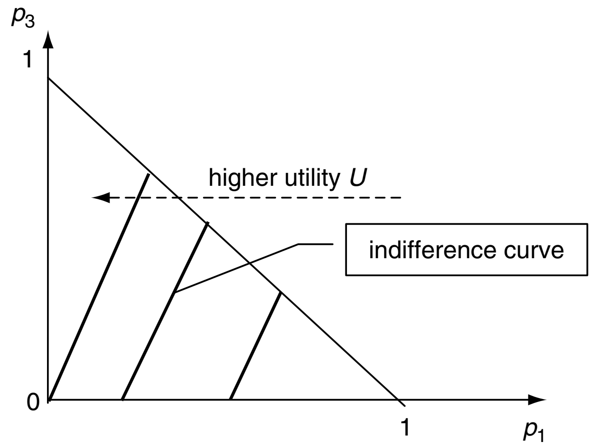
it is, by definition, preferences are linear in probability. That is, an indifference curve at u_0 is linear in (p_1, p_3) space:

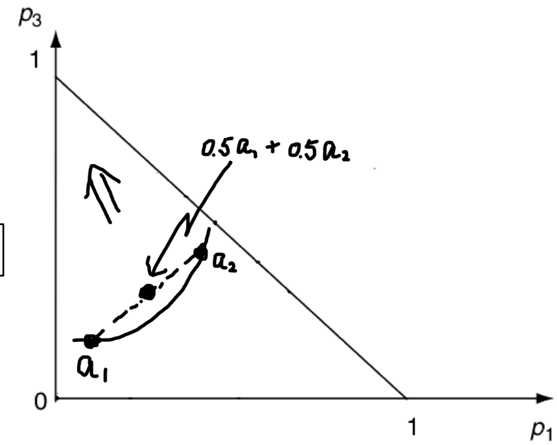
$$\begin{aligned}
& u_0 = \mathbb{E}[U(x)] \\
\Leftrightarrow \ & u_0 = p_1 U(x_1) + (1 - p_1 - p_3)U(x_2) + p_3 U(x_3) \\
\Leftrightarrow \ & p_3 = m + n p_1
\end{aligned}$$

where

$$m = \frac{u_0 - U(x_2)}{U(x_3) - U(x_2)}, \quad n = \frac{U(x_2) - U(x_1)}{U(x_3) - U(x_2)}.$$

As usual, u_0 determines the position of the curve. The following shows three curves based on different u_0.





Source: Chavas (2004)

Next, to see how it arises from the independence assumption, let's consider $a_1 \sim a_2$, i.e. a_1 and a_2 are on some indifference curve. Remember

$$a_1 \sim a_2 \quad \Leftrightarrow \quad a_1 \succsim a_2 \text{ and } a_2 \succsim a_1.$$

Then, if some indifference curve is not a straight line, a convex combination is preferred such as

$$\alpha a_1 + (1-\alpha) a_2 \succ a_1 \tag{1}$$

for $\alpha \in (0,1)$. But, the independence assumption requires

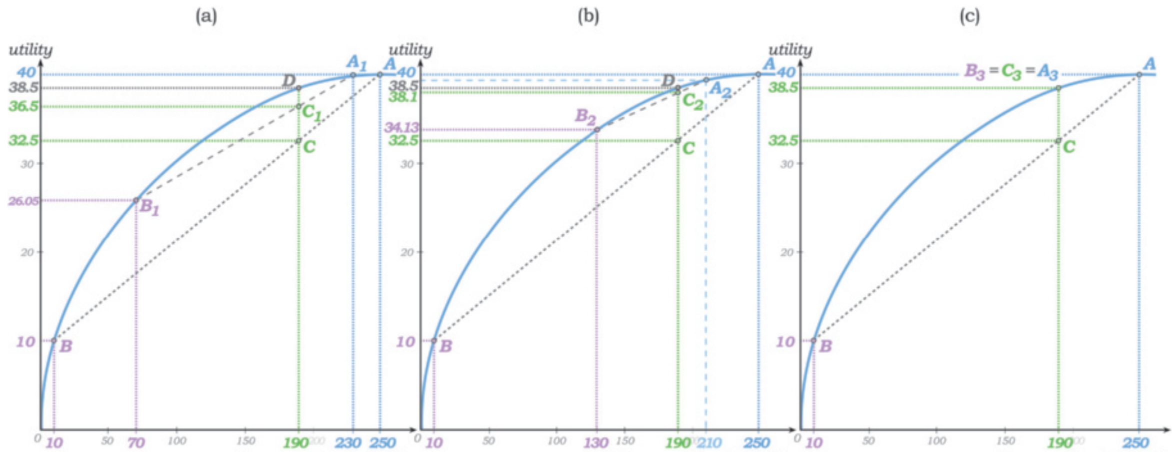
$$\begin{aligned}
& a_1 \sim a_2 \\
\Leftrightarrow \quad & a_1 \succsim a_2 \quad \text{and} \quad a_2 \succsim a_1 \\
\Leftrightarrow \quad & \alpha a_1 + (1-\alpha) a_1 \succsim \alpha a_1 + (1-\alpha) a_2 \quad \text{and} \quad \alpha a_1 + (1-\alpha) a_2 \succsim \alpha a_1 + (1-\alpha) a_1 \\
\Leftrightarrow \quad & a_1 \succsim \alpha a_1 + (1-\alpha) a_2 \quad \text{and} \quad \alpha a_1 + (1-\alpha) a_2 \succsim a_1 \\
\Leftrightarrow \quad & \alpha a_1 + (1-\alpha) a_2 \sim a_1,
\end{aligned}$$

which is a contradiction to (1). Thus, we must have the linearity. For more details, see section 6.B in Mas-Colell, Whinston, and Green (1995).

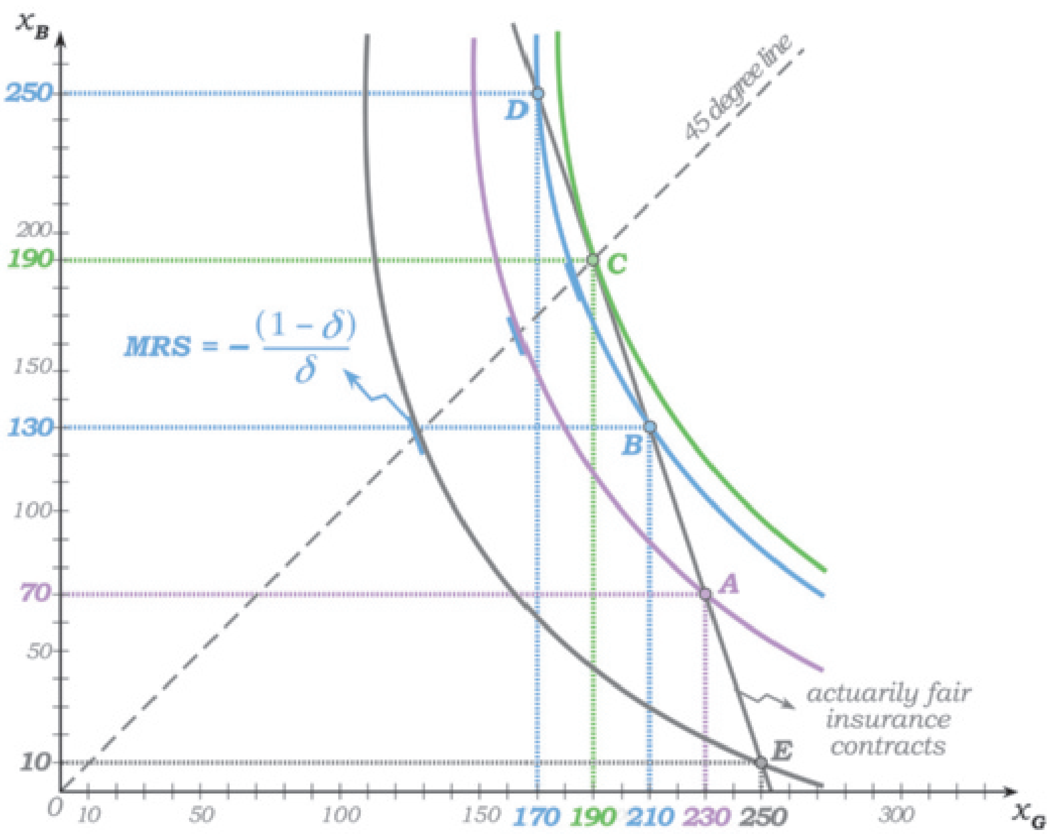
State preference approach

If the linearity of preference in probability causes so much trouble, a natural question is "Can we do any risk analysis without probabilities?"

Let's imagine that you face a risky situation (x) involving a good state (\$250) and a bad state (\$10) with the probability of the bad state equal to $\delta = 0.25$. So, $\mathbb{E}(x) = 190$. Suppose that you are risk averse and interested in buying an insurance that provides a unit benefit of \$80 for a unit premium of \$20. How many units would you buy?



Source: Nechyba (2011)



Source: Nechyba (2011)

It is a lot like the standard choice setting: ordinal utility, convex upper-contour set, endowment, optimality at the tangency of an indifference curve and the constraint, etc. Goods you consume here are called state-contingent commodities. The state preference approach is a basis for general equilibrium analysis under uncertainty e.g. Radner equilibrium. One thing to keep in mind is that each commodity is state-contingent and you experience only one of the states and therefore consume only one of commodity, which is different from the standard consumption bundle (e.g. 250 apples and 10 oranges).