Dynamic optimization under risk

The problem is

$$\max_{x_1,\ldots,x_T} \mathbb{E}[U(w, x_1, \ldots, x_T, e)], \tag{1}$$

where w is an initial wealth and e is a random variable. As we have seen throughout this course, decisions under risk often mean that we directly choose e among alternative random variables or alter e through decisions, i.e. e is a function of $x_1, \ldots, x_T$. Despite the extra stochasticity, the basic framework is the same as deterministic cases covered in the last week. A key difference is how we define choice and state. Under risk, choices are basically random variables and state is something that affects random variables (i.e. information) and thereby affecting choices.

Since decisions $x_1, \ldots, x_T$ here are made sequentially over time, we may re-write it as

$$\max_{x_1} \mathbb{E}_1[\max_{x_2} \mathbb{E}_2[\ldots \max_{x_T} \mathbb{E}_T[U(w, x_1, \ldots, x_T, e)] \ldots]], \tag{2}$$

where $\mathbb{E}_t$ denote expectation taken with respect to $f_t(e)$, the subjective probability distribution of e based on the information available at t. If there is no learning over time (i.e. the information is unchanged and $f_1(e) = \cdots = f_T(e)$), (1) and (2) are identical. We are interested in situations where

today's decision x_t affects tomorrow's information $f_{t+1}(e)$.

In other words, available information at t defines the state at t. If obtaining information is costly, there is a tradeoff involved in making decisions $x_1, \ldots, x_T$. Hence, dynamic optimization.

Two-period case

Let's study a two-period case as a building block. With $T = 2$, the problem is

$$\max_{x_1,x_2} \mathbb{E}[U(w, x_1, x_2, e)],$$

and if there is learning, it is

$$\max_{x_1} \mathbb{E}_1[\max_{x_2} \mathbb{E}_2[U(w, x_1, x_2, e)|x_1]]. \tag{3}$$

Suppose that some information is obtained after making x_1 decision by receiving a signal s_1 (s_1 is a function of x_1). To model a "signal" of e or their dependency, we usually see s_1 as a realization of a random variable S_1 and are interested in the conditional probability of s_1 given e or the likelihood. When s_1 tells nothing about e, they are independent and $f_2(e|s_1) \neq f_2(e)$.

To solve (3) by backward induction, we first solve

$$\max_{x_2} \mathbb{E}_2[U(w, x_1, x_2, e)|x_1]$$

conditional on x_1, in particular, using the information $s_1(x_1)$ if any. Imagine a specific case where $x_1 \in \{Buy, NotBuy\}$, i.e whether to buy the information s_1 by paying B dollars. We are indifferent to buying if

$$\max_{x_2} \mathbb{E}_1[U(w, x_1, x_2, e)] = \max_{x_2} \mathbb{E}_2[U(w - B, x_1, x_2, e)|x_1].$$

Or, we buy if we pay less. Notice $B = 0$ if $x_1 = NotBuy$. Also,

$$\max_{x_2} \mathbb{E}_1[U(w, x_1, x_2, e)] = \max_{x_2} \mathbb{E}_1[U(w, x_1, x_2, e)|x_1],$$

because we are making x_1 and x_2 decisions simultaneously at $t = 1$. We may say B is the value of the information s_1.

After getting

$$x_2^*(x_1) = \operatorname*{argmax}_{x_2} \mathbb{E}_2[U(w - B, x_1, x_2, e)|x_1],$$

we substitute it in (3) and proceed to solving

$$\max_{x_1} \mathbb{E}_1[U(w, x_1, x_2^*(x_1), e)].$$