Multi-product firm under uncertainty

Setup

Consider a firm producing m products under:

Output	$y = (y_1, \dots, y_m)^T$
Net return	$p = (p_1, \dots, p_m)^T$
Profit	$\pi = p^T y$
Utility function	$U(\pi)$

Notice that p denotes the net return, which means that p reflects both output prices and production costs per unit. Here, we assume only output price uncertainty, so only p are random variables (aka. a random vector). The problem is, then, to find

$$y^* = \operatorname*{argmax}_{y \in \mathcal{Y}} \mathbb{E}[U(\pi)],$$

where $\mathcal{Y}$ is a feasible set of production. To facilitate the analysis, we restrict our attention to the effects of correlation between p_i and p_j for all $i, j \in \{1, \dots, m\}$ under the mean-variance model. Recall the mean-variance model (Chapter 6), which assumes

$$\mathbb{E}[U(\pi)] = W(\mathbb{E}[\pi], \text{Var}[\pi]).$$

To find an optimal y^*, we need to write $\mathbb{E}[\pi]$ and $\text{Var}[\pi]$ as functions of y.

$$\mathbb{E}[\pi] = \mathbb{E}[p^T y] = \mathbb{E}\left[\sum_{i=1}^{m} p_i y_i\right] = \sum_{i=1}^{m} \mathbb{E}[p_i] y_i = \sum_{i=1}^{m} \mu_i y_i = \mu^T y,$$

where $\mu = (\mu_1, \dots, \mu_m)$ is the expected value of p, and

$$\text{Var}[\pi] = \text{Var}[p^T y] = y^T \text{Var}[p] y = y^T \Sigma y,$$

where A is the covariance matrix of p

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{bmatrix}, \quad \sigma_{ij} = \text{Cov}(p_i, p_j).$$

Remember that $U(\cdot)$ takes a scalar value π, so $\mathbb{E}[\pi]$ and $\text{Var}[\pi]$ are scalar whereas p is a vector and Σ is a matrix. Having written $\mathbb{E}[\pi]$ and $\text{Var}[\pi]$ as functions of y, the problem is now to find

$$y^* = \operatorname*{argmax}_{y \in \mathcal{Y}} W(\mathbb{E}[\pi], \text{Var}[\pi]). \tag{1}$$

E-V frontier

If the form of expected utility $W(\cdot)$ is known, we may directly solve (1). If not, we may take a different approach. Recall that under the mean-variance model with risk aversion assumed, for given $M = \mu^T y$, it is optimal to minimize $\mathrm{Var}[\pi]$.

$$\begin{aligned} y^+(M) &= \operatorname*{argmin}_{y \in \mathcal{Y}} \mathrm{Var}[\pi] \text{ s.t. } M = \mu^T y \\ &= \operatorname*{argmin}_{y \in \mathcal{Y}} y^T \Sigma y \text{ s.t. } M = \mu^T y. \end{aligned} \tag{2}$$

This is a quadratic programing (Boyd & Vandenberghe, 2004), i.e. a quadratic objective function with linear constraints. So, there exists a solution. Indeed, we may set up the Lagrangian:

$$\mathcal{L}(y, \lambda) = y^T \Sigma y - \lambda(\mu^T y - M)$$

and examine the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y} = 2\Sigma y - \lambda \mu = 0 \\ \mu^T y = M \end{aligned}$$

or

$$\begin{pmatrix} 2\Sigma & -\mu \\ \mu^T & 0 \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ M \end{pmatrix} \tag{3}$$

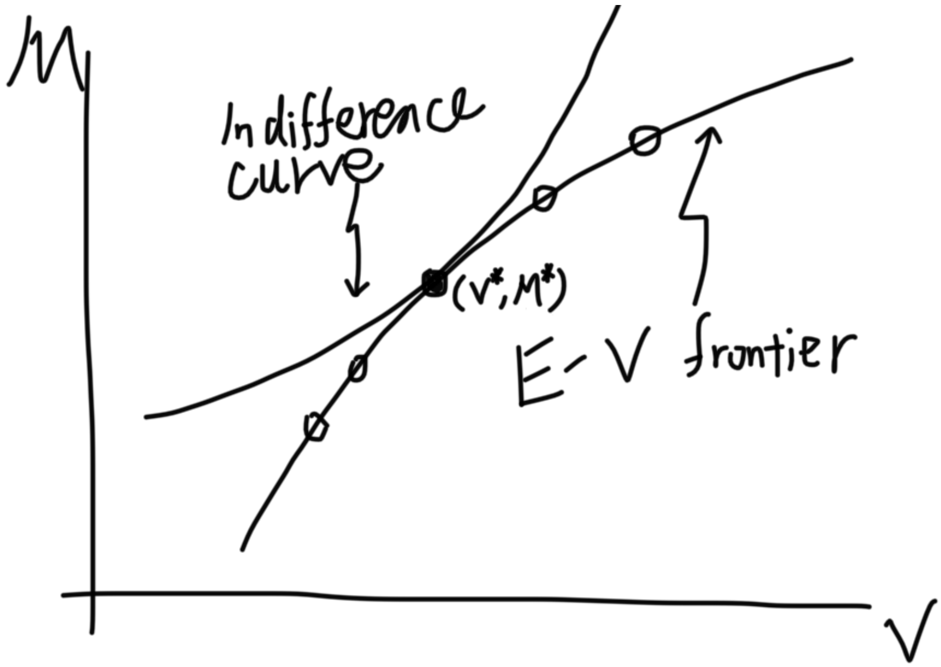
Note that the matrix on the left-hand side is not necessarily invertible.

The E-V frontier is a set

$$\left\{ (V, M) \in \mathbb{R}^2_+ : V = \min_{y \in \mathcal{Y}} y^T \Sigma y \ \text{ s.t. } M = \mu^T y \right\}.$$

We can also see it as a graph and plot it with V on the horizontal axis and M on the vertical axis.

With the E-V frontier in hand, we may ask a firm manager which point (V, M) on the curve he/she prefers. The point chosen (V^*, M^*) tells us a solution $y^* = y^+(M^*)$. Even without the knowledge of $W(\cdot)$ or indifference curves, picking (V^*, M^*) implies that the unknown indifference curve is tangent at that point.



Diversification

Let's look at the effects of correlation under $m = 2$. Let ρ be the correlation between p_1 and p_2. Then, rewrite the covariance matrix as follows

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Notice, for brevity, $\sigma_1^2 = \text{Var}[p_1] = \sigma_{11}$ and $\sigma_2^2 = \text{Var}[p_2] = \sigma_{22}$ in the previous notation. Observe,

$$\begin{aligned} \text{Var}[\pi] &= y^T \Sigma y \\ &= \sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 + 2\rho\sigma_1\sigma_2 y_2 y_2. \end{aligned}$$

Clearly, if $\rho = -1$,

$$\begin{aligned} \text{Var}[\pi] &= \sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 - 2\sigma_1\sigma_2 y_2 y_2 \\ &= (\sigma_1 y_1 - \sigma_2 y_2)^2, \end{aligned}$$

which is minimized to 0 by choosing

$$\begin{aligned} & \sigma_1 y_1 = \sigma_2 y_2 \\ \Leftrightarrow \quad & \sigma_1 y_1 = \sigma_2 \frac{M - \mu_1 y_1}{\mu_2} \\ \Leftrightarrow \quad & y_1 = \frac{M\sigma_2}{\mu_2\sigma_1 + \mu_1\sigma_2}. \end{aligned}$$

Hence, with the perfect negative correlation ($\rho = -1$), we can completely eliminate the risk regardless of M. That is, the E-V frontier is a vertical line at $V = 0$.

How about $\rho = 0$? Notice that, with $\rho = 0$, Σ is diagonal and positive definite. So, the matrix in (3) is invertible.

$$\begin{aligned}
\begin{pmatrix} y \\ \lambda \end{pmatrix} &= \begin{pmatrix} 2\Sigma & -\mu \\ \mu^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ M \end{pmatrix} \\
&= \frac{1}{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2} \begin{pmatrix} \frac{1}{2}\mu_2^2 & -\frac{1}{2}\mu_1\mu_2 & \mu_1\sigma_2^2 \\ -\frac{1}{2}\mu_1\mu_2 & -\frac{1}{2}\sigma_1^2 & \mu_2\sigma_1^2 \\ -\mu_1\sigma_2^2 & -\mu_2\sigma_1^2 & 2\sigma_1^2\sigma_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ M \end{pmatrix},
\end{aligned}$$

which implies

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{M}{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2} \begin{pmatrix} \mu_1\sigma_2^2 \\ \mu_2\sigma_1^2 \end{pmatrix}.$$

As a result,

$$\begin{aligned}
V &= \mathrm{Var}[\pi] \\
&= y^T\Sigma y \\
&= \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \end{pmatrix} \left(\frac{M}{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2} \right)^2 \begin{pmatrix} \mu_1^2\sigma_2^4 \\ \mu_2^2\sigma_1^4 \end{pmatrix} \\
&= \frac{\mu_1^2\sigma_1^2\sigma_2^4 + \mu_2^2\sigma_1^4\sigma_2^2}{(\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2)^2} M^2.
\end{aligned}$$

Hence, the E-V frontier is $M = \alpha\sqrt{V}$ where $\alpha > 0$, which means that it is an upward straight line if M is plotted against standard deviation.